Int. J. Heat Mass Transfer. Vol. 14, pp. 1677–1690. Pergamon Press 1971. Printed in Great Britain

ANALYSIS OF HEAT TRANSFER FOR COMPRESSIBLE FLOW IN TWO-DIMENSIONAL POROUS MEDIA

MARVIN E. GOLDSTEIN and ROBERT SIEGEL

Lewis Research Center, Cleveland, Ohio, U.S.A.

(*Received* 19 *October* 1970 *and in revised form* 9 *February* 1971)

Abstract—Gas from a reservoir at constant pressure and temperature is forced through a two-dimensional porous region. The surface through which the gas exits is at a specified uniform temperature and pressure. The local gas and solid matrix temperatures are assumed equal. General solutions for the local temperature and pressure in the porous medium are found as a function of a potential. This potential can be determined by solving Laplace's equation in the porous region for a simple set of boundary conditions, and the temperature and pressure will then be known functions of position. Because of the nature of the boundary conditions it is particularly convenient to solve Laplace's equation by conformal mapping. By using this technique some illustrative heat and mass flow results were calculated for a porous wall with a step in thickness, a wall supplied with gas through periodic slots, and an eccentric annular region.

NOMENCLATURE

A,	thickness ratio of step porous wall or dimensionless half width of reservoir opening for supported porous wall, a/h_r;
a,	largest thickness of step porous wall or half width of reservoir opening for supported porous wall;
B,	dimensionless half period of periodic supported porous wall (Figs. 4 and 7);
C_p,	specific heat at constant pressure;
F,	normal (incomplete) elliptic integral of first kind;
g,	intermediate mapping parameter for eccentric annular porous region;
h_r,	general reference length of porous region;
K,	complete elliptic integral of the first kind $K(k)$;
K',	defined by $K'(k) \equiv K(\sqrt{1 - k^2})$;
k,	modulus of elliptic integral;
k_1, k_2,	mapping parameters defined by equations (48);
k_m,	effective thermal conductivity of porous region;
L_s,	dimensionless arc length along inside cylinder of eccentric porous region, l_s/h_r;
l_s,	arc length along inside cylinder of eccentric porous region;
$\hat{n}$,	unit outward drawn normal;
P,	pressure ratio, p/p_0;
p,	pressure;
$\boldsymbol{Q}$,	dimensionless energy flux vector, $\boldsymbol{q}h_r/k_m t_\infty$;
$\boldsymbol{q}$,	energy flux vector;
R,	gas constant;
R_0,	dimensionless radius of cylinder;
S_l, S_r, S, S_0,	surfaces bounding porous region in dimensionless coordinates;
s_l, s_r,	left and right hand surfaces bounding porous region;
s,	upper (or inner) surface bounding porous region;
s_0,	lower (or outer) surface bounding porous region;
T,	temperature ratio, t/t_∞;
t,	absolute temperature;
U,	intermediate mapping variable;

$\boldsymbol{u}$, velocity vector;
V, intermediate mapping variable;
W, complex potential, $\psi + i\phi$;
X, dimensionless coordinate in physical plane, x/h_r;
x, coordinate in physical plane;
X_1, X_2, dimensionless coordinates to locate inner eccentric cylinder;
Y, dimensionless coordinate in physical plane, y/h_r;
y, coordinate in physical plane;
Z, complex physical coordinate, $X + iY$;
η, parametric variable for equation (49);
κ, permeability of porous material;
λ, parameter $\rho_\infty C_p \kappa p_0/(2\mu k_m)$;
μ, gas viscosity;
ξ, real part of ω;
ρ, gas density;
Φ, energy flux potential;
Φ_s, constant surface potential determined by equation (32);
ϕ, dimensionless energy flux potential, Φ/Φ_s;
ψ, intermediate variable, $(\ln R_0)\psi$;
ψ, harmonic conjugate of ϕ, $\mathcal{R}e W$;
ω, intermediate mapping variable.

Subscripts
0, on lower (or outer) boundary s_0;
∞, in coolant reservoir;
s, on upper (or inner) boundary s;
l, on left hand boundary s_l;
r, on right hand boundary s_r.

INTRODUCTION

A METHOD for extending the use of a metallic structural material to higher temperature applications is to provide transpiration cooling. The metal is made in a porous form and coolant is forced through it from a reservoir toward the boundary exposed to the high temperature source. Some possible applications are for cooling turbine blades, rocket nozzles, arc electrodes, and portions of surfaces during either high speed flight or reentry into the Earth's atmosphere.

The energy equation governing the temperature distribution in a porous medium contains the velocity distribution of the coolant within the matrix material. This velocity is generally a complicated function of position thereby complicating the energy equation. Moreover, when the cooling fluid is a gas the energy equation and the momentum equation are coupled and the problem becomes nonlinear. In the present paper it will be shown that a large class of two-dimensional porous cooling problems where the fluid is a gas can be reduced to solving Laplace's equation and therefore can be solved by standard techniques such as conformal mapping. Previous solutions to porous cooling problems dealing with two-dimensions and/or compressible flows were limited to one-dimensional (plane, cylindrical and spherical) compressible flows [1], two-dimensional incompressible flows [2–4] and a numerical solution for compressible two-dimensional flow [5].

It will be assumed that the thermal resistance between the fluid and matrix material is small so that the local fluid and matrix temperatures are equal. As a consequence, a single energy equation can be written that includes the heat transport by conduction in the matrix material and by coolant convection. One boundary of the porous region is in contact with the coolant reservoir which is at constant pressure and temperature. The boundary through which the coolant exits is at constant pressure and has a specified uniform temperature. The heat and mass flows in the porous region are governed by the energy conservation equation, the equation of continuity, Darcy's law, and the perfect gas law. The solution of these equations is obtained by using the energy conservation equation to define a potential that satisfies Laplace's equation in the porous region and a simpler set of boundary conditions. The remaining equations are then solved to obtain general expressions for the temperature and pressure distributions in terms of this potential. Upon solving Laplace's equation in any specific region to obtain an expression for this potential in terms of the physical coordinates, the temperature and pressure become known functions of position.

The porous region maps into a simple strip or rectangular geometry in the complex potential plane, and consequently conformal mapping is a convenient technique for solving Laplace's equation. By using this method, some illustrative results are obtained for a porous wall with a step in thickness, a wall with gas supplied through periodic slots on one boundary, and an eccentric annular region.

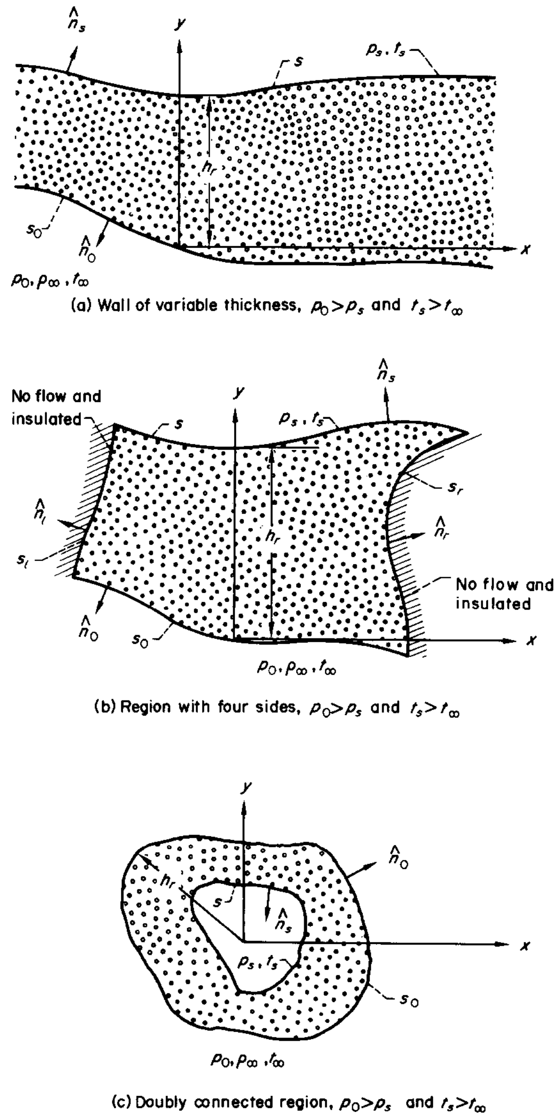
ANALYSIS

Physical system being analyzed

The types of geometries of the porous media that will be considered are shown schematically in cross section in Fig. 1. It is assumed that no changes occur in the direction perpendicular to the x–y plane so that the situations are two dimensional. Figure 1(a) is a long wall of arbitrarily varying thickness; Fig. 1(b) is a finite region where two opposing sides are insulated and have no fluid flowing across them; Fig. 1(c) is a doubly connected region. The symbol h_r denotes a characteristic dimension for each geometry.

The lower [or outer in the case of Fig. 1(c)] surface s_0 of any of these porous regions has an outward-drawn unit normal vector $\hat{n}_0$. The upper [or inner in the case shown in Fig. 1(c)] surface s has an outward-drawn unit normal vector $\hat{n}_s$. In the case of the region shown in Fig. 1(b) the left and right hand surfaces s_l and s_r, respectively, have unit outward drawn normals $\hat{n}_l$ and $\hat{n}_r$, respectively. Let k_m denote the effective thermal conductivity (based on the entire cross sectional area) of the porous medium and let κ denote its permeability.

An ideal gas whose density and pressure are denoted by ρ and p respectively is flowing through the porous region. The specific heat at constant pressure C_p and for simplicity the viscosity μ of the gas will be taken as constant. The use of a representative average viscosity value (over the range of gas temperatures involved) is not expected to change the surface heat and mass fluxes in any qualitative way. In addition, it will be assumed that the thermal conductivity of the gas is very small compared with k_m and that for a given pressure drop across the porous region the pore size is sufficiently small so that Darcy's law governs the



(a) Wall of variable thickness, $p_0 > p_s$ and $t_s > t_\infty$

(b) Region with four sides, $p_0 > p_s$ and $t_s > t_\infty$

(c) Doubly connected region, $p_0 > p_s$ and $t_s > t_\infty$

FIG. 1. Porous configurations being analyzed.

flow of the gas through this region. Let μ denote the Darcy velocity (local volume flow divided by entire cross-sectional area rather than by pore cross sectional area) of the gas.

For each geometry the region below [or outside in the case shown in Fig. 1(c)] the porous material is a reservoir which is maintained at constant pressure and temperature p_0 and t_∞, and therefore at constant density ρ_∞. The upper (or inner) surface of the porous region is at a constant temperature t_s and the region above (or inside) the porous region is maintained at the constant pressure p_s. We suppose that $p_0 > p_s$ so that the gas flows from the reservoir through the porous wall and out through the top (or inner) surface. Since p_0 and p_s are both constant, the gas velocity at both the upper and lower wall surfaces will be in a direction which is perpendicular to these surfaces. The temperature t_s at the upper (or inner) surface is constant and larger than t_∞ so that heat will flow by conduction from the upper (or inner) surface towards the reservoir. The following analysis applies if the direction of both the heat and mass flows are reversed but some details of the analysis must be changed if the direction of only one of these flows is reversed.

Governing equations

It is assumed that the thermal communication between the fluid and the porous matrix is sufficiently good, so that the local fluid temperature will be approximately equal to the local matrix temperature. We denote this common temperature by t. It will be assumed that the pore size is very small compared with any overall dimension of the porous region. When these assumptions are made, the flow of heat and mass within the porous region are governed by the following equations:

$$\nabla . \boldsymbol{q} = 0 \qquad \text{(Conservation of energy)} \tag{1}$$

$$\boldsymbol{q} \equiv -k_m \nabla t + \rho \boldsymbol{u} C_p t \tag{2}$$

$$\nabla . (\rho \boldsymbol{u}) = 0 \qquad \text{(Conservation of mass)} \tag{3}$$

$$\boldsymbol{u} = -\frac{\kappa}{\mu} \nabla p \qquad \text{(Darcy's law)} \tag{4}$$

$$p = \rho R t \qquad \text{(Ideal gas law).} \tag{5}$$

The vector $\boldsymbol{q}$ defined in equation (2) is the net energy flux vector with the kinetic energy term neglected. (The neglect of the kinetic energy is consistent with the approximation that Darcy's law holds.) Darcy's law as given by equation (4) applies for compressible flow (see [6] and [7]). In the following we shall assume for simplicity that the thermal conductivity k_m and the permeability κ are constants.

Boundary conditions

As the gas in the reservoir approaches the porous region the gas accelerates to the entering velocity (the pressure change associated with this velocity change is negligible compared to the pressure drop through the porous region), and temperature rises from the reservoir temperature t_∞ to the surface temperature t_0 of the porous medium which is an *a priori* unknown variable along s_0. Since the thermal conductivity of the gas is assumed to be much less than k_m, the thickness of the gas layer over which this temperature rise takes place is very small compared with the porous region thickness provided the flow is not very small. We can, therefore, assume that this thermal layer is locally one dimensional. Since the velocity is perpendicular to s_0 there is no flow along this thermal layer. Hence, by applying an energy balance to the thermal layer the boundary conditions are

$$\left.\begin{array}{c} k_m \hat{n}_0 . \nabla t = \rho C_p (t - t_\infty) \hat{n}_0 . \boldsymbol{u} \\ p = p_0 = \text{constant} \end{array}\right\} \quad \text{for} \quad (x, y) \text{ on } s_0. \tag{6}$$

On the upper (or inner) surface a uniform temperature is specified so that the boundary conditions are

$$\left.\begin{array}{c} t = t_s = \text{constant} \\ p = p_s = \text{constant} \end{array}\right\} \text{for} \quad (x, y) \text{ on } s. \tag{7}$$

In the case of the bounded region shown in Fig. 1(b) the additional boundary conditions along the sides of the region are obtained by noting that since there is no flow of heat or mass across the side surfaces s_l and s_r, the normal derivative of both the pressure and the temperature must vanish at these surfaces. Hence,

$$\left.\begin{aligned} \hat{n}_l \cdot \nabla t &= 0 \\ \hat{n}_l \cdot \nabla p &= 0 \end{aligned}\right\} \text{ for } \quad (x, y) \text{ on } s_l \tag{8a}$$

$$\left.\begin{aligned} \hat{n}_r \cdot \nabla t &= 0 \\ \hat{n}_r \cdot \nabla p &= 0 \end{aligned}\right\} \text{ for } \quad (x, y) \text{ on } s_r. \tag{8b}$$

Dimensionless form of equations

It is now convenient to introduce the following dimensionless quantities

$$\lambda \equiv \frac{\rho_\infty C_p \kappa p_0}{2\mu k_m} = \frac{C_p \kappa p_0^2}{2\mu k_m R t_\infty} \tag{9}$$

$$\left.\begin{aligned} X &= x/h_r \\ Y &= y/h_r \\ T &= t/t_\infty \\ T_s &= t_s/t_\infty \\ P &= p/p_0 \\ P_s &= p_s/p_0 \\ \boldsymbol{Q} &= \boldsymbol{q} h_r/k_m t_\infty \\ \tilde{\nabla} &= \hat{\imath}\frac{\partial}{\partial X} + \hat{\jmath}\frac{\partial}{\partial Y}. \end{aligned}\right\} \tag{10}$$

Upon substituting equations (4) and (5) into equations (2) and (3) to eliminate $\boldsymbol{u}$ and ρ, and using the dimensionless quantities, we find that equations (1)–(3) can be written as

$$\tilde{\nabla} \cdot \boldsymbol{Q} = 0 \tag{11}$$

$$\boldsymbol{Q} = -\tilde{\nabla}(T + \lambda P^2) \tag{12}$$

$$\tilde{\nabla}^2(P^2) = (1/T)\,\tilde{\nabla} T \cdot \tilde{\nabla}(P^2). \tag{13}$$

Upon using equations (4) and (5) and the dimensionless quantities in the boundary conditions (6)–(8b) we find the following convenient dimensionless forms:

$$\left.\begin{aligned} \frac{T}{T-1}\hat{n}_0 \cdot \tilde{\nabla} T + \lambda \hat{n}_0 \cdot \tilde{\nabla}(P^2) &= 0 \\ P &= 1 \end{aligned}\right\} \text{ for } \quad (X, Y) \text{ on } S_0 \tag{14}$$

$$\left.\begin{aligned} T &= T_s \\ T + \lambda P^2 &= T_s + \lambda P_s^2 \end{aligned}\right\} \text{ for } \quad (X, Y) \text{ on } S \tag{15}$$

$$\left.\begin{aligned} \hat{n}_l \cdot \tilde{\nabla} T &= 0 \\ \hat{n}_l \cdot \tilde{\nabla}(T + \lambda P^2) &= 0 \end{aligned}\right\} \text{ for } \quad (X, Y) \text{ on } S_l \tag{16a}$$

$$\left.\begin{aligned} \hat{n}_r \cdot \tilde{\nabla} T &= 0 \\ \hat{n}_r \cdot \tilde{\nabla}(T + \lambda P^2) &= 0 \end{aligned}\right\} \text{ for } \quad (X, Y) \text{ on } S_r. \tag{16b}$$

Solution for T *and* P *in terms of a potential*

Now, equation (12) shows that the energy flux is equal to the gradient of a potential. Therefore, we put

$$\Phi \equiv T - C_0 + \lambda(P^2 - 1) \tag{17}$$

where C_0 is a constant to be specified subsequently. Equation (12) then gives

$$\boldsymbol{Q} = -\tilde{\nabla}\Phi$$

so equation (11) yields

$$\tilde{\nabla}^2 \Phi = 0. \tag{18}$$

Thus, the heat flux potential Φ satisfies Laplace's equation. The boundary conditions (15)–(16b) supply boundary conditions for Φ directly but the boundary condition (14) will only give a boundary condition which couples Φ with T. Hence, equation (18) must be solved simultaneously with equation (13) in order to find the solution to the boundary value problem posed above. We shall now show however that this boundary value problem can be solved by assuming *a priori* that T is a function of Φ only, that is,

$$T(X, Y) = T[\Phi(X, Y)]. \tag{19}$$

Then equation (17) shows that P is a function of Φ only. Hence,

$$\tilde{\nabla}T = \frac{\mathrm{d}T}{\mathrm{d}\Phi}\tilde{\nabla}\Phi \tag{20}$$

and therefore, equation (17) shows that

$$\lambda\tilde{\nabla}(P^2) = \tilde{\nabla}\Phi - \tilde{\nabla}T = \left(1 - \frac{\mathrm{d}T}{\mathrm{d}\Phi}\right)\tilde{\nabla}\Phi. \tag{21}$$

Equations (18) and (21) imply

$$\lambda\tilde{\nabla}^2(P^2) = -\frac{\mathrm{d}^2T}{\mathrm{d}\Phi^2}|\tilde{\nabla}\Phi|^2. \tag{22}$$

Substituting equations (20)–(22) into equation (13) shows that

$$\left[\frac{\mathrm{d}^2T}{\mathrm{d}\Phi^2} + \frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}\Phi}\left(1 - \frac{\mathrm{d}T}{\mathrm{d}\Phi}\right)\right]|\tilde{\nabla}\Phi|^2 = 0.$$

Hence equation (13) will be satisfied if we put

$$\frac{\mathrm{d}^2T}{\mathrm{d}\Phi^2} + \frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}\Phi}\left(1 - \frac{\mathrm{d}T}{\mathrm{d}\Phi}\right) = 0$$

or equivalently

$$\frac{\mathrm{d}}{\mathrm{d}\Phi}\left[\frac{1}{T}\left(\frac{\mathrm{d}T}{\mathrm{d}\Phi} - 1\right)\right] = 0.$$

Upon integration we find that

$$T = \frac{1}{C_1} + C_2\,\mathrm{e}^{-C_1\Phi} \tag{23}$$

where C_1 and C_2 are constants.

These constants, are evaluated by using the boundary conditions. Thus, substituting equations (20) and (21) into the first boundary condition (14) yields

$$\left(1 + \frac{1}{T-1}\frac{\mathrm{d}T}{\mathrm{d}\Phi}\right)\hat{n}_0 \,.\, \tilde{\nabla}\Phi = 0 \quad \text{for} \quad (X, Y) \text{ on } S_0.$$

Substituting equation (23) into this result yields

$$(1 - C_1)\frac{T}{T-1}\hat{n}_0 \,.\, \tilde{\nabla}\Phi = 0 \quad \text{for} \quad (X, Y) \text{ on } S_0.$$

Hence, the first boundary condition (14) is satisfied if we put

$$C_1 = 1.$$

Equation (23) then becomes

$$T = 1 + C_2\,\mathrm{e}^{-\Phi}. \tag{24}$$

When equation (24) is substituted into equation (17) and the results used in the second boundary condition (14) we find

$$\Phi = 1 - C_0 + C_2\,\mathrm{e}^{-\Phi} \quad \text{for} \quad (X, Y) \text{ on } S_0$$

so that the second boundary condition (14) is satisfied when $\Phi(X, Y) =$ constant for (X, Y) on S_0. Since C_0 is arbitrary we can put

$$C_0 = 1 + C_2 \tag{25}$$

to obtain

$$\Phi(X, Y) = 0 \quad \text{for} \quad (X, Y) \text{ on } S_0. \tag{26}$$

Consideration of the boundary conditions (15) on S shows that it is convenient to define a constant Φ_s by

$$\Phi_s \equiv T_s - 1 - C_2 + \lambda(P_s^2 - 1). \tag{27}$$

Then it follows from equations (17), (24) and (25) that the boundary conditions (15) will be satisfied if we put

$$\Phi(X, Y) = \Phi_s \quad \text{for} \quad (X, Y) \text{ on } S \tag{28}$$

$$C_2 = (T_s - 1)\,\mathrm{e}^{\Phi_s}. \tag{29}$$

Equations (29) and (25) are used to eliminate C_2 and C_0 from equations (24) and (17) giving the desired general temperature and pressure distributions by the following functions of Φ:

$$T = 1 + (T_s - 1)\,\mathrm{e}^{\Phi_s - \Phi} \tag{30}$$

$$\lambda(P^2 - 1) = (T_s - 1)\,\mathrm{e}^{\Phi_s}(1 - \mathrm{e}^{-\Phi}) + \Phi. \tag{31}$$

The constant Φ_s is determined by the following relation from equations (29) and (27),

$$\Phi_s = (T_s - 1)(1 - e^{\Phi_s}) + \lambda(P_s^2 - 1). \quad (32)$$

Notice that since $P_s < 1$ and $T_s > 1$, the right hand side of equation (32) will always be negative if Φ_s is positive. Hence, for the condition of interest here, Φ_s must be negative if it is to satisfy equation (32). More generally, since $T_s > 1$ and P^2 is a decreasing function of distance in going from S_0 to S it follows from equation (31) that Φ is also a decreasing function of distance in going from S_0 to S.

For the case shown in Fig. 1(b) the boundary conditions (16a) and (16b) must also be satisfied. It follows from equations (20) and (17) that these conditions will be satisfied if we put

$$\hat{n}_l \, . \, \tilde{\nabla}\Phi = 0 \quad \text{for} \quad (X, Y) \text{ on } S_l \quad (33a)$$

$$\hat{n}_r \, . \, \tilde{\nabla}\Phi = 0 \quad \text{for} \quad (X, Y) \text{ on } S_r. \quad (33b)$$

Thus, the solution to the differential equations (11)–(13) that satisfied the boundary conditions (14)–(16b) is given by equations (30)–(32). The function Φ in these equations is determined uniquely by equation (18) and the boundary conditions (26), (28) and in addition, for the case shown in Fig. 1(b) equations (33a) and (33b).

Equations to determine potential in terms of physical coordinates

Equations (30) and (31) express the temperature and pressure distributions as explicit functions of a potential. Thus, once the potential is determined as a function of the physical coordinates, then so are the temperature and pressure. In the relations for the potential function it is convenient to simplify the boundary conditions by defining a normalized potential Φ by

$$\phi(X, Y) \equiv \frac{\Phi(X, Y)}{\Phi_s} \, . \quad (34)$$

Then it follows from equations (18), (26), (28), (33a) and (33b) that ϕ can be obtained by solving Laplace's equation

$$\tilde{\nabla}^2\phi = 0 \quad (35a)$$

with the boundary conditions

$$\phi(X, Y) = 0 \quad \text{for} \quad (X, Y) \text{ on } S_0 \quad (35b)$$

$$\phi(X, Y) = 1 \quad \text{for} \quad (X, Y) \text{ on } S \quad (35c)$$

$$\left.\begin{aligned} \hat{n}_l \, . \, \tilde{\nabla}\phi = 0 \quad \text{for} \quad (X, Y) \text{ on } S_l \\ \hat{n}_r \, . \, \tilde{\nabla}\phi = 0 \quad \text{for} \quad (X, Y) \text{ on } S_r. \end{aligned}\right\} \quad (36)$$

Relations for heat and mass flux at surface of porous medium

Two quantities of important practical interest are the heat flux conducted into the solid and the mass flux leaving the porous region at the surface S.

The heat flux conducted into the porous material at the upper (or inner) surface is given by

$$q_s = k_m \hat{n}_s \, . \, \nabla t \quad \text{for} \quad (x, y) \text{ on } s.$$

Introducing dimensionless quantities and using equation (30) gives

$$\frac{q_s h_r}{k_m(t_s - t_\infty)} = - \hat{n}_s \, . \, \tilde{\nabla}\Phi \quad \text{for} \quad (X, Y) \text{ on } S.$$

But Φ is constant on S. Hence (since Φ is a decreasing function of distance when going from S_0 to S)

$$\hat{n}_s = \frac{-\tilde{\nabla}\Phi}{|\tilde{\nabla}\Phi|} \quad (37)$$

and upon using equation (34)

$$\frac{q_s h_r}{|\Phi_s| k_m(t_s - t_\infty)} = |\tilde{\nabla}\phi| \quad \text{for} \quad (X, Y) \text{ on } S. \quad (38)$$

The mass flux at the upper or inner surface is found from equations (4), (5), (9) and (10) to be

$$\rho \boldsymbol{u} \, . \, \hat{n}_s = - \frac{\kappa}{\mu} \rho \nabla p \, . \, \hat{n}_s = - \frac{k_m}{C_p h_r} \frac{1}{T} \tilde{\nabla}(\lambda P^2) \, . \, \hat{n}_s$$
$$\text{for} \quad (X, Y) \text{ on } S.$$

Upon using equations (31), (37) and (34) this becomes

$$h_r C_p \left(\frac{\rho \boldsymbol{u} . \hat{n}_s}{k_m |\Phi_s|} \right) = |\tilde{\nabla} \phi| \quad \text{for} \quad (X, Y) \text{ on } S. \tag{39}$$

Use of conformal mapping to obtain ϕ(X, Y)

The nature of the boundary conditions (35b) and (35c), makes it particularly convenient to use conformal mapping to relate ϕ to X and Y. Since ϕ is a solution to Laplace's equation, there exists a harmonic function ψ and an analytic function W of the complex variable

$$Z = X + iY \tag{40}$$

such that

$$W = \psi + i\phi. \tag{41}$$

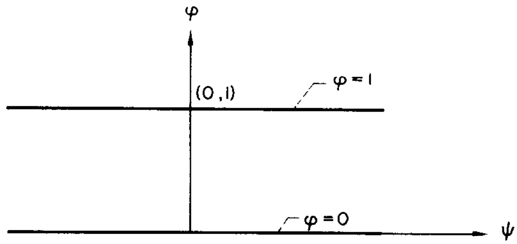
Physically the change in ψ between any two points in the physical plane is proportional to the net energy flow across any curve joining those two points. Hence, for the case shown in Fig. 1(a), ψ must vary between $-\infty$ and $+\infty$ as X varies between $-\infty$ and $+\infty$. For the case shown in Fig. 1(b) it follows from the Cauchy–Riemann equations that the boundary conditions (36) are equivalent to requiring that ψ be equal to a constant on S_l and S_r. Hence, in this case

$$\begin{aligned} \mathcal{R}e W &= \text{constant for } Z \text{ on } S_l \\ \mathcal{R}e W &= \text{constant for } Z \text{ on } S_r. \end{aligned} \tag{42}$$

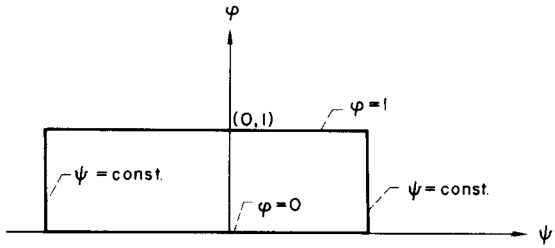
Equations (35b) and (c) show that in all cases

$$\begin{aligned} \mathcal{I}m W = 0 \quad &\text{for} \quad Z \text{ on } S_0 \\ \mathcal{I}m W = 1 \quad &\text{for} \quad Z \text{ on } S. \end{aligned} \tag{43}$$

Hence, in the case shown schematically in Fig. 1(a) the mapping $Z \to W$ transforms the interior of the porous medium into the infinite strip shown schematically in Fig. 2(a) in the W-plane. For the case shown in Fig. 1(b) the mapping $Z \to W$ transforms the interior of the porous region into the rectangular region in the W-plane shown in Fig. 2(b). In order to obtain the mapping into the potential plane for the case shown in Fig. 1(c) recall that every sufficiently



(a) Region corresponding to Fig. 1 (a)



(b) Region corresponding to Figs. 1 (b) and (c)

FIG. 2. Potential planes ($W = \psi + i\varphi$) corresponding to porous regions.

smooth doubly connected region can be mapped into an annulus. Let the annulus be in the U-plane and have a radial cut in it. Then the mapping $W = i \ln U$ maps the slit annulus in the U-plane into the rectangular region shown in Fig. 2(b) and hence, the mapping $Z \to W$ transforms the interior of the porous region shown in Fig. 1(c) also into the rectangular region in the W-plane shown in Fig. 2(b).

To obtain expressions for the surface heat and mass fluxes we use the following well-known relation given in [10] (p. 182),

$$|\tilde{\nabla} \phi| = \left| \frac{\mathrm{d}W}{\mathrm{d}Z} \right|.$$

Then equations (38) and (39) become

$$\frac{q_s h_r}{|\Phi_s| k_m (t_s - t_\infty)} = h_r C_p \left(\frac{\rho \boldsymbol{u} . \hat{n}_s}{k_m |\Phi_s|} \right) = \left| \frac{\mathrm{d}W}{\mathrm{d}Z} \right| \quad \text{for} \quad (X, Y) \text{ on } S. \tag{44}$$

Thus, once the mapping $W \to Z$ is determined, which transforms the porous region in the physical plane into the rectangle or strip in the potential plane and which depends only on the geometry of the porous region, the surface heat and mass fluxes can be calculated from equation (44).

APPLICATION OF GENERAL SOLUTION

Step porous wall

As an example solution for a porous region of the type shown in Fig. 1(a) consider the porous wall with a step change in cross section shown in Fig. 3. In this figure all lengths have been made dimensionless by dividing by the smaller thickness. The mapping which transforms this region into the unit strip of Fig. 2(a) is given parametrically in terms of a complex variable ω by [8].

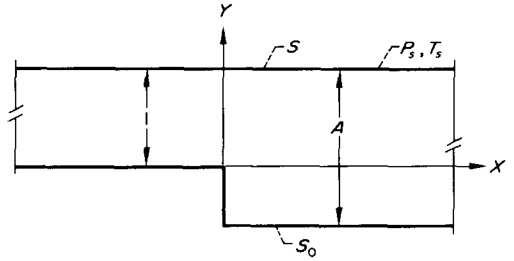


FIG. 3. Step porous wall in dimensionless physical plane.

$$Z = \frac{1}{\pi}\left[A \ln\left(\frac{\omega + 1}{\omega - 1}\right) - \ln\left(\frac{\omega + A}{\omega - A}\right)\right]$$
$$\omega = \left(\frac{e^{\pi W} - A^2}{e^{\pi W} - 1}\right)^{1/2} \tag{45}$$

where $1 \leqslant \omega \leqslant A$ for Z on S.

Upon differentiating equation (45) we find that

$$\frac{dW}{dZ} = \frac{\omega}{A}.$$

Let $\xi = \mathscr{R}e\,\omega$. Then for Z on S, $|dW/dZ|$ is given parametrically as a function of the distance X_s along the upper surface by

$$\left.\begin{aligned} X_S &= \frac{1}{\pi}A\left[\ln\left(\frac{\xi + 1}{\xi - 1}\right)\right. \\ &\quad \left. - \ln\left(\frac{A + \xi}{A - \xi}\right)\right] \\ \left|\frac{dW}{dZ}\right| &= \frac{\xi}{A} \end{aligned}\right\} 1 \leqslant \xi \leqslant A. \tag{46}$$

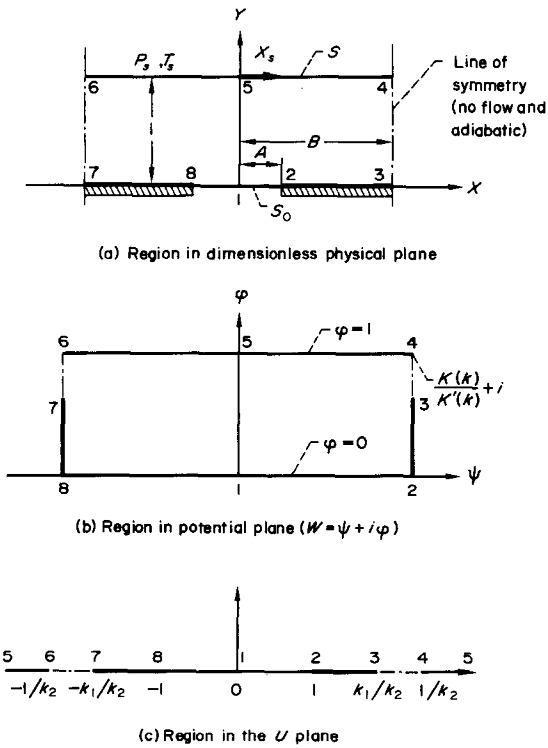


FIG. 4. Porous wall with coolant supplied through periodic openings.

Porous wall with gas supplied through periodic slots

As an example of the porous region of the type shown in Fig. 1(b) consider the section of wall shown in Fig. 4(a). This may represent a section of a long porous wall with a periodic distribution of openings on the reservoir side. The remainder of the boundary on the reservoir side could correspond to supports which are frequently necessary when porous structures are used. All lengths are normalized by the thickness of the wall. Since both the porous region in the physical plane and the corresponding region in the W-plane [Fig. 4(b)] are rectangles they can both be mapped into the upper half U-plane shown in Fig. 4(c) by elliptic functions [9]. The mappings which transform corresponding points into corresponding points (as indicated in Fig. 4) are

$$\left.\begin{aligned} U &= sn[WK'(k_2), k_2] \\ \frac{k_2}{k_1} U &= sn(ZK'(k_1), k_1) \end{aligned}\right\} \quad (47)$$

where

$$\left.\begin{aligned} B &= \frac{K(k_1)}{K'(k_1)} \\ k_2 &= k_1\, sn[AK'(k_1), k_1] \end{aligned}\right\} \quad (48)$$

K is the complete elliptic integral of the first kind and

$$K'(k) = K(\sqrt{1 - k^2}).$$

Upon differentiating equations (47) we find

$$\frac{\mathrm{d}W}{\mathrm{d}Z} = \frac{k_1}{k_2}\frac{K'(k_1)}{K'(k_2)}\left(\frac{1 - \left(\frac{k_2}{k_1}U\right)^2}{1 - U^2}\right)^{\frac{1}{2}}.$$

As a consequence of these mappings, we find that for Z on S, $|\mathrm{d}W/\mathrm{d}Z|$ is given as a parametric function of distance X_s along the upper surface in Fig. 4(a) by

$$\left.\begin{aligned} \left|\frac{\mathrm{d}W}{\mathrm{d}Z}\right| &= \frac{K'(k_1)}{K'(k_2)}\sqrt{\frac{1 - \eta^2 k_1^2}{1 - \eta^2 k_2^2}} \quad (49) \\ X_s &= \frac{F(\sin^{-1}\eta, k_1)}{K'(k_1)} \end{aligned}\right\} \; 0 \leqslant \eta \leqslant 1$$

where F is the normal elliptic integral of the first kind and we have put

$$\eta = (k_2\, \mathscr{R}e U)^{-1}.$$

Eccentric annular region

An example of a porous region of the type shown in Fig. 1(c) is the region between the two eccentric circular cylinders shown in Fig. 5(a). All lengths are made dimensionless by the radius of the large cylinder. It is shown in [9], p. 370, and [10], p. 287, that this region is mapped into the annular region in the V-plane in the manner

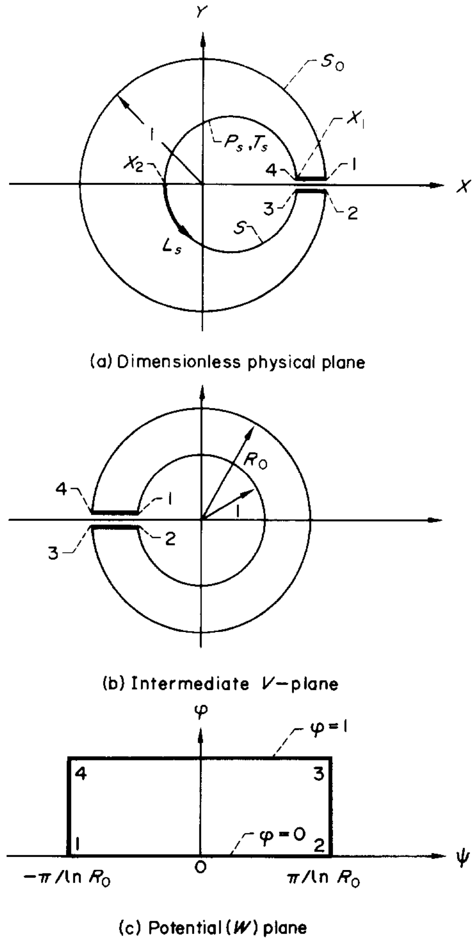


FIG. 5. Porous eccentric annular region.

indicated in Figs 5(a) and (b) by

$$V = \frac{Z - g}{gZ - 1}$$

where

$$g = \frac{1 + X_1 X_2 + \sqrt{(1 - X_1^2)(1 - X_2^2)}}{X_1 + X_2}$$

and the outer radius R_0, shown in Fig. 5(b), is given by

$$R_0 = \frac{1 - X_1 X_2 + \sqrt{(1 - X_1^2)(1 - X_2^2)}}{X_1 - X_2}.$$

The annular region in the V-plane is mapped into the rectangular region in the W-plane in the manner indicated in Fig. 5(c) by

$$W = i\frac{\ln V}{\ln R_0} = i\ln\left(\frac{Z-g}{gZ-1}\right)\Big/\ln R_0. \quad (50)$$

As a result of these relations we find that $|dW/dZ|$ with Z on S; is given as a parametric function of dimensionless distance L_s [see Fig. 5(a)] along the inner circle by

$$\left.\begin{aligned} L_s &= (X_1 - X_2)\left\{\tan^{-1}\left[\left(\frac{1-X_2^2}{1-X_1^2}\right)^{\frac{1}{2}}\tan\frac{\tilde{\psi}}{2}\right]\right\} \\ \left|\frac{dW}{dZ}\right| &= \frac{1 - 2\sigma\cos\tilde{\psi} + \sigma^2}{(\ln R_0)\dfrac{(X_1 - X_2)}{2}(\sigma^2 - 1)} \end{aligned}\right\} -\pi \leqslant \tilde{\psi} \leqslant \pi \quad (51)$$

where σ is defined by

$$\sigma \equiv \frac{(1-X_2^2)^{\frac{1}{2}} + (1-X_1^2)^{\frac{1}{2}}}{(1-X_2^2)^{\frac{1}{2}} - (1-X_1^2)^{\frac{1}{2}}}$$

and we have put $\tilde{\psi} = (\ln R_0)\psi$.

These results can now be used with equations (44) to compute the heat flux into the porous region and the mass flux through the porous region at the surface s through which the gas is exiting from the region.

RESULTS AND DISCUSSION

The main purpose of this paper has been to develop a general solution for a certain class of two-dimensional compressible porous cooling problems. This class of problems corresponds to the situation where gas from a reservoir at constant pressure and temperature is forced through the porous region and exits at a boundary maintained at constant temperature into a region of constant pressure. The solution is obtained by using the equation of conservation of energy to define a potential that satisfies Laplace's equation within the porous region and that satisfies certain simple boundary conditions on the boundary of the region. The remaining equations that govern the problem can then be solved to provide general solutions for the pressure and temperature distributions explicitly in terms of this potential. Therefore,

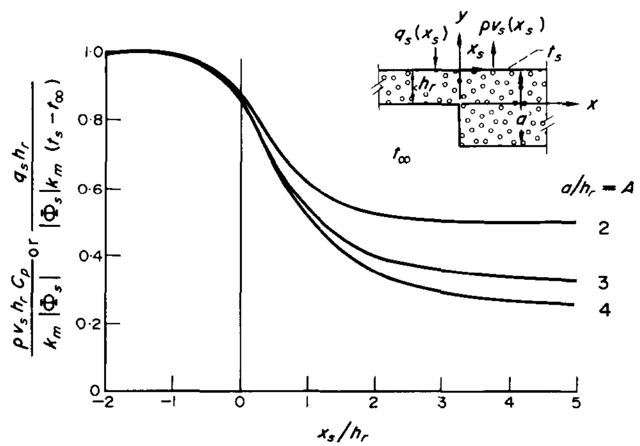


FIG. 6. Dimensionless mass flux and heat flux at surface of step porous wall.

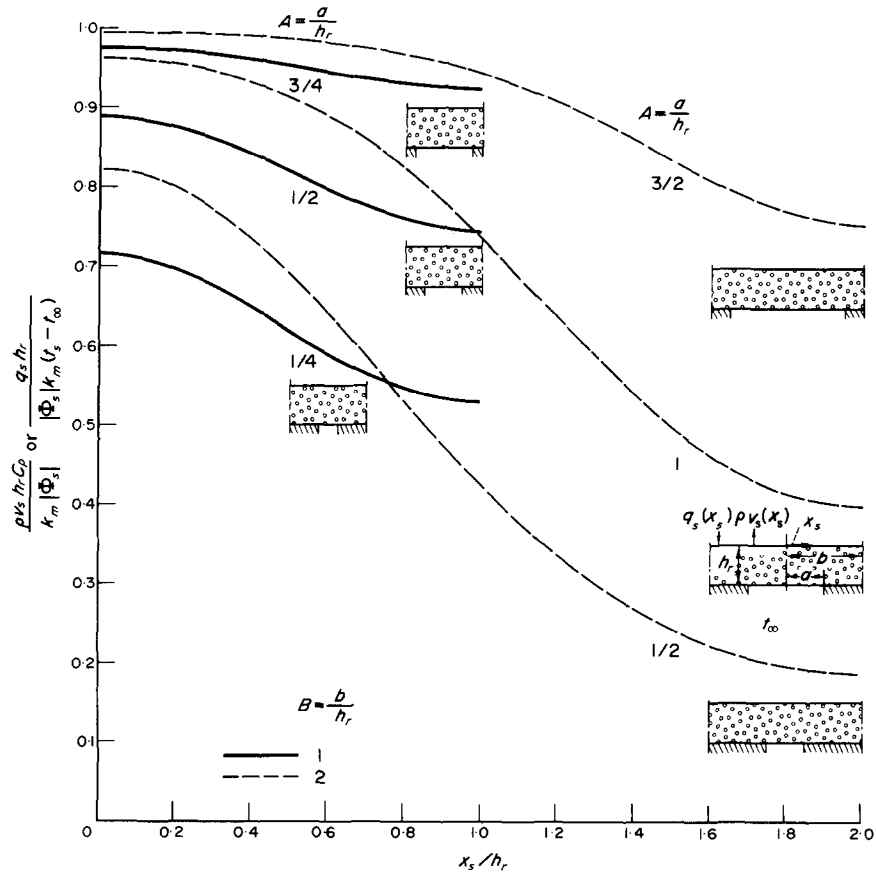


FIG. 7. Dimensionless mass flux and heat flux at surface of porous wall with coolant introduced through periodic slots.

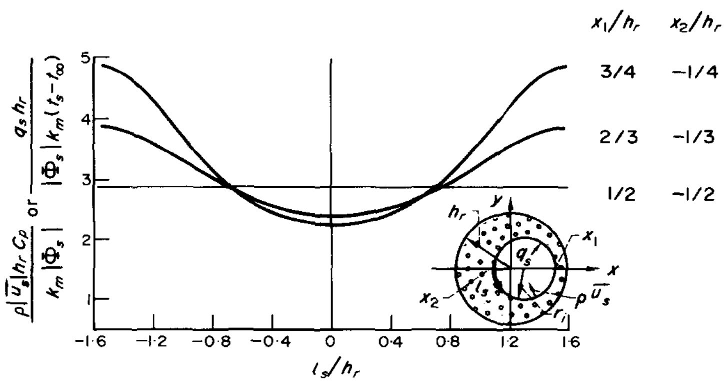


FIG. 8. Dimensionless mass flux and heat flux along the inside of an eccentric porous annulus (radius of inner circle $r_i = h_r/2$).

by solving Laplace's equation in any specific region to obtain an expression for this potential in terms of the physical coordinates, the temperature and pressure are determinable functions of position.

Several illustrative examples were worked out in detail by using conformal mapping to obtain the required solutions to Laplace's equation. These results are shown in Figs. 6–8. In each figure a dimensionless mass flux leaving the porous region and a dimensionless heat flux at the surface are plotted as functions of position along the surface. The density in the exit mass flux is that corresponding to the imposed exit pressure and surface temperature. The quantity Φ_s in the dimensionless fluxes is obtained from equation (32).

Figure 6 gives results for walls having various step changes in their thickness as designated by the parameter A. As a special case it is evident from the left end of the curves that for a wall having a uniform thickness h_r the results are given by

$$\frac{\rho v_s h_r C_p}{k_m|\Phi_s|} = \frac{q_s h_r}{|\Phi_s|\, k_m(t_s - t_\infty)} = 1.$$

Consequently, for the step wall, the dimensionless mass and heat fluxes vary from unity to h_r/a, where a is the thickness of the thick region. The curves show that the two-dimensional effects are primarily confined to within one thickness of the thin portion of the wall to the left of the step ($x_s/h_r = -1$) and one thickness of the thick portion to the right of the step ($x_s/a = 1$ or $x_s/h_r = A$).

Figure 7 shows results for a wall that is of uniform thickness but has its lower boundary only partially exposed to the coolant reservoir. For small blockage, that is when the opening a is close to the width b, the curves go toward unity, which is the result for an unobstructed plane wall. As expected the highest velocities and heat fluxes are at the exit locations opposite the center of the opening.

Results for eccentric annular porous regions are shown in Fig. 8 for various eccentricities and for the case where the radius of the inner circle is one-half that of the outer circle. As would be expected the largest flows occur where the wall is thin. For the concentric case the solutions from equations (44) and (51) reduce to

$$\frac{\rho \boldsymbol{u}_s h_r C_p}{k_m|\Phi_s|} = \frac{q_s h_r}{|\Phi_s|\, k_m(t_s - t_\infty)} = \frac{h_r/r_i}{\ln (h_r/r_i)}.$$

The three sets of results shown in Figs. 6–8 serve to demonstrate the type of results that can be obtained from the general two-dimensional compressible solution developed in this paper.

REFERENCES

1. D. R. BLAND, Mathematical theory of the flow of a gas in a porous solid and of the associated temperature distributions, *Proc. R. Soc.* **221A**, 1–28 (1954).
2. J. C. Y. KOH and E. DEL CASAL, Heat and mass flow through porous matrices for transpiration cooling, Proceedings of the 1965 Heat Transfer and Fluid Mechanics Institute, edited by A. F. CHARWAT, pp. 263–281. Stanford University Press, Stanford, Calif. (1965).
3. R. SIEGEL and M. GOLDSTEIN, Analytical method for steady state heat transfer in two-dimensional porous media, NASA TN D-5878 (July 1970).
4. P. J. SCHNEIDER, R. E. MAUER and M. G. STRAPP, Two-dimensional transpiration cooling, AIAA Paper 69–96, AIAA 7th Aerospace Sciences Meeting, New York (Jan. 1969).
5. E. P. DEL CASAL, The effects of multidimensional flow through porous matrices in mass transfer cooling, AIAA Paper 69–149, AIAA 7th Aerospace Sciences Meeting, New York (Jan. 1969).
6. M. MUSKAT, *The Flow of Homogeneous Fluids Through Porous Media.* J. W. Edwards, Ann Arbor, Mich. (1946).
7. A. E. SCHEIDEGGER, *The Physics of Flow Through Porous Media.* Macmillan, New York (1957).
8. L. M. MILNE-THOMSON, *Theoretical Hydrodynamics,* 2nd ed. Macmillan, New York (1950).
9. GINO MORETTI, *Functions of a Complex Variable,* p. 359. Prentice-Hall, New York (1964).
10. R. V. CHURCHILL, *Complex Variables and Applications,* 2nd ed. McGraw-Hill, New York (1960).

ANALYSE DU TRANSFERT THERMIQUE POUR UN ECOULEMENT COMPRESSIBLE DANS UN MILIEU POREUX BIDIMENSIONNEL

Résumé—On force à travers une région poreuse bidimensionnelle un gaz provenant d'un réservoir à pression et température constantes. La surface de sortie est à une température et à une pression uniformes données. Les températures locales du gaz et de la matrice solide sont supposées égales. On trouve que les solutions générales pour la température et la pression locales dans le milieu poreux sont fonctions d'un potentiel. Ce potentiel peut être déterminé par la résolution de l'équation de Laplace dans la région poreuse pour un ensemble simple de conditions aux limites et on connaît alors la température et la pression comme des fonctions de point. A cause de la nature des conditions aux limites, il est assez facile de résoudre l'équation de Laplace par une transformation conforme. Pour illustrer cette technique quelques résultats massique et thermique sont données pour une paroi poreuse avec un échelon dans l'épaisseur une paroi alimentée en gaz à travers des fentes périodiques et une région annulaire excentrique.

ANALYSIS DES WÄRMEÜBERGANGS BEI KOMPRESSIBLER STRÖMUNG IN ZWEIDIMENSIONALEN PORÖSEN MEDIEN

Zusammenfassung— Gas wird von einem Behälter konstanten Druckes und konstanter Temperatur durch eine zweidimensionale, poröse Zone geleitet. Die Oberfläche, durch die das Gas austritt, wird auf ganz bestimmten gleichförmigen Werten von Temperatur und Druck gehalten. Die lokalen Werte der Gas- und Feststofftemperatur werden als gleich angenommen. Allgemeine Lösungen für die lokalen Temperatur- und Druckverteilungen im porösen Medium werden als Funktion eines Potentials gefunden. Dieses Potential kann durch die Lösung der Laplace-Gleichung in der porösen Zone für einige einfache Randbedingungen bestimmt werden. Temperatur und Druck sind dann bekannte Funktionen des Ortes. Wegen der Art der Randbedingungen ist es sehr bequem, die Laplace-Gleichung durch konforme Abbildung zu lösen. Mit dieser Technik wurden einige aufschlußreiche Ergebnisse für den Wärme- und Massenstrom errechnet für eine poröse Wand mit stufenförmig geänderter Dicke, eine Wand mit regelmäßigen Gasdurchlaßschlitzen und einer exzentrischen Kreisringzone.

АНАЛИЗ ТЕПЛООБМЕНА ДЛЯ СЛУЧАЯ НЕСЖИМАЕМОГО ТЕЧЕНИЯ В ДВУМЕРНЫХ ПОРИСТЫХ СРЕДАХ

Аннотация—Газ из резервуара при постоянных давлении и температуре под давлением просачивается через двумерный пористый участок. Поверхность, через которую выходит газ, находится при заданной однородной температуре и давлении. Локальные температуры газа и твёрдого скелета принимаются одинаковыми. Решения в общем виде для локальной температуры и давления в пористой среде находятся как функция потенциала. Этот потенциал можно определить из решения уравнения Лапласа в пористом участке для пористой системы граничных условий, после чего температура и давление будут известными функциями положения. Благодаря характеру граничных условий практически удобно решать уравнение Лапласа с помощью конформного отображения. С помощью этой методики рассчитаны некоторые наглядные результаты для потока тепла и массы в случае пористой стенки со скачком толщины, стенки с подачей газа через периодически расположенные щели и эксцентричного кольцевого участка.